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Abstract Scattering Theory

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Abstract

The asymptotic condition is formulated for a system whose theory is more general than quantum mechanics. Its logic $\mathscr{L}$ forms an orthocomplemented weakly modular σ-lattice. The set of states $\mathscr{S}$, consisting of all the probability measures on $\mathscr{L}$, is endowed with the most suitable metric physically, called here the natural one. In this space it is proved that the asymptotic condition implies the existence of two convex automorphisms $\Omega_\pm$ of $\mathscr{S}$ which we call the wave-automorphisms. From these the S-automorphism $\Omega_-^{-1}\Omega_+$ is defined and corresponds to the scattering operator in conventional quantum theory.

1. *Introduction*

A formulation of abstract scattering theory is presented here in a generalised sense both physically and mathematically. There is no mention of physical space, particles or waves. The system is supposed to obey a theory more general than quantum mechanics, in the sense that its logic $\mathscr{L}$ is a *generalised* one. That is to say that the set of propositions $\mathscr{L}$ (Jauch, 1968a) is an orthocomplemented, weakly modular σ-lattice (Varadarajan, 1968). The set $\mathscr{S}$ of *proper states* consists of all the probability measures on $\mathscr{L}$.

The starting point is a precise mathematical formulation of the so-called *asymptotic condition.* This roughly says that the initial (remote past) data and the final (distant future) data of an undisturbed system and those in the presence of a perturbation are nearly indistinguishable. This statement cannot be made precise without an appropriate topology on $\mathscr{S}$. We use the most important one physically, which in fact makes $\mathscr{S}$ a metric space; the metric is called here the natural one. The dynamical groups are one-parameter groups of convex automorphisms of $\mathscr{S}$, and at first sight it seems difficult to transcribe the asymptotic condition to the existence of the wave automorphisms in this abstract formalism. In quantum theory this is obviously more straightforward, since the dynamical group is a one-parameter group of unitary operators on a separable Hilbert space (Jauch, 1958). The aim of this article is to show exactly how this difficulty can be overcome. From this we prove the existence of the wave-automorphisms $\Omega_\pm$ which are isometric convex automorphisms of $\mathscr{S}$ and correspond in conventional quantum theory to Møller's wave-operators. From this the scattering automorphisms $S = \Omega_-^{-1}\Omega_+$ is defined and corresponds to the S-matrix in quantum mechanics.

2. *The Banach Space of States*

We use standard lattice theoretic notation (Varadarajan, 1968). From now on $\mathscr{L}$ will denote an orthocomplemented, weakly modular σ-lattice, which we call a *generalised logic*. Axiomatically it is very reasonable to assume that the proposition system is a generalised logic. It is much more general than the quantum one; the latter satisfies extra conditions for which many physicists do not see physical significance. The set $\mathscr{S}$ of *proper states* consists of all the probability measures on $\mathscr{L}$.

Essential to the formulation of the asymptotic condition is the choice of an appropriate topology on $\mathscr{S}$. In conventional quantum theory there are several metric topologies all giving the same asymptotic condition but of course different topologies (Jauch *et al.*, 1968b). The most important one physically in the generalised formalism $(\mathscr{L}, \mathscr{S})$ considered here is that defined by the metric

$$\rho(p,q) = \sup\{|p(a) - q(a)| : a \in \mathscr{L}\} \qquad (p, q \in \mathscr{S})$$

The proof that this is a metric is quite trivial. We shall call it the *natural metric*.

Symmetries and dynamical groups are directly related to the *convex automorphisms* of $\mathscr{S}$. Such an automorphism is a one-one mapping $A: p \to Ap$ of $\mathscr{S}$ onto itself such that for any sequence (c_n) of non-negative numbers with

$$\sum_n c_n = 1$$

and any sequence (p_n) in $\mathscr{S}$,

$$A \sum_n c_n p_n = \sum_n c_n A p_n$$

The set of all convex automorphisms of $\mathscr{S}$ forms a group $\mathrm{Aut}(\mathscr{S})$, with identity denoted by E. The difficulties encountered when handling representations of topological groups by $\mathrm{Aut}(\mathscr{S})$ are removed by the following steps.

We define a *generalised state* or *signed measure* on $\mathscr{L}$ to be a real-valued function p on $\mathscr{L}$ satisfying

(i) $-\infty < p(a) < \infty \qquad (a \in \mathscr{L})$,
(ii) $p(\emptyset) = 0$,
(iii) for any disjoint sequence (a_n) in $\mathscr{L}$

$$p\left(\bigvee_n a_n\right) = \sum_n p(a_n)$$

The set of all generalised states will be denoted throughout by X. It is clearly a real linear space containing $\mathscr{S}$ as a convex subset. A generalised state which is not a proper one does not have any physical significance. The set X is employed merely for derivation of rigorous mathematical results on $(\mathscr{L}, \mathscr{S})$ and their automorphisms.

For $p \in X$, we define its *upper* and *lower variations* at $a \in \mathscr{L}$ by

$$\bar{p}(a) = \sup\{p(x) : x < a\} \qquad \text{and} \qquad \underline{p}(a) = \sup\{-p(x) : x < a\}$$

respectively. The *total variation* at a is $|p|(a) = \bar{p}(a) + \underline{p}(a)$. Important properties of signed measures over σ-algebras such as the boundedness of variation, the Jordan decomposition $p = \bar{p} - \underline{p}$ and that $\bar{p}$, $\underline{p}$ and $|p|$ are themselves measures, also hold for generalised states (Kronfli, 1969).

The following two theorems are important for the later development in this paper.

Theorem 2.1

The set X of all generalised states on $\mathscr{L}$ is a real Banach space with norm

$$\|p\| = |p|(I) \qquad (p \in X)$$

where I is the identity in $\mathscr{L}$. Furthermore,

(i) $\mathscr{S}$ is a closed convex subset of X, with the norm $\|\cdot\|$ inducing the natural metric ρ,
(ii) span $(\mathscr{S})$ over the reals equals X.

Theorem 2.2

If $A \in \mathrm{Aut}(\mathscr{S})$, then it can be extended uniquely to a linear operator $\tilde{A}$ on X such that

(i) $\|\tilde{A}\| = 1$,
(ii) $^{-1}$ exists and equals $\widetilde{A^{-1}}$.

These theorems have been stated and proved by Kronfli (1969). The role played by X is similar to that played by Hilbert space in conventional quantum theory. For this reason we can call X the *Banach space of states*. The convex automorphisms of $\mathscr{S}$ can now be easily handled. We shall use the same symbol for an automorphism and its operator extension of Theorem 2.2.

3. *The Asymptotic Condition*

We define the *dynamical group* $\mathscr{D}$ of the perturbed system to be a mapping $t \to U(t)$ of the real line into $\mathrm{Aut}(\mathscr{S})$ such that

(i) $U(0) = E$,
(ii) $U(t_1)\,U(t_2) = U(t_1 + t_2) \qquad (t_1, t_2 \in R)$,

(iii) $\mathscr{D}$ is ρ-strongly continuous in R, i.e., for each $t_0 \in R$

$$\lim_{t \to t_0} \rho(U(t)p, U(t_0)p) = 0$$

for all $p \in \mathscr{S}$.

The *unperturbed dynamical group* $\mathscr{D}_0$ is also a mapping $t \to U_0(t)$ of the real line into Aut$(\mathscr{S})$ satisfying conditions (i)–(iii) above.

Consider a system with perturbed and unperturbed dynamical groups $\mathscr{D}$ and $\mathscr{D}_0$, respectively. We say that $\mathscr{D}$ satisfies the *asymptotic condition* (A) relative to $\mathscr{D}_0$ if for every proper state $p \in \mathscr{S}$, there exists a unique pair of states $p^{\pm} \in \mathscr{S}$ such that

$$\lim_{t \to \mp\infty} \rho(U(t)p^{\pm}, U_0(t)p) = 0 \tag{A}$$

It is now straightforward to prove the existence of Møller's wave-automorphisms.

Theorem 3.1

If the dynamical group $\mathscr{D}$ of the scattering system satisfies the asymptotic condition (A) relative to $\mathscr{D}_0$, then the limits

$$\Omega_{\pm} = s - \lim_{t \to \mp\infty} U(-t)\, U_0(t)$$

exist in X and define convex automorphisms of $\mathscr{S}$.

Proof: By the asymptotic condition, for any $p \in \mathscr{S}$ there exist a unique pair $p^{\pm} \in \mathscr{S}$ such that

$$\| U(t)p^{\pm} - U_0(t)p \| \to 0 \qquad (t \to \mp\infty)$$

since the norm of X induces the natural metric according to Theorem 2.1. Noting that $\| U(t) \| = 1$ from Theorem 2.2, we have

$$\begin{aligned} \| p^{\pm} - U(-t)\, U_0(t)p \| &= \| U(-t)\, [U(t)p^{\pm} - U_0(t)p] \| \\ &\leqslant \| U(-t) \| \, \| U(t)p^{\pm} - U_0(t)p \| \\ &= \| U(t)p^{\pm} - U_0(t)p \| \to 0 \qquad (t \to \mp\infty) \end{aligned}$$

Hence

$$\lim_{t \to \mp\infty} U(-t)\, U_0(t)p$$

exist for all $p \in \mathscr{S}$. Since $X = \text{span}(\mathscr{S})$, then

$$\Omega_{\pm} = s - \lim_{t \to \mp\infty} U(-t)\, U_0(t)$$

exist on X. Similarly

$$\Omega_{\pm}^{-1} = s - \lim_{t \to \mp\infty} U_0(-t)\, U(t)$$

also exist and it is elementary to show that $\Omega_\pm$ are convex automorphisms of $\mathscr{S}$.

We call $\Omega_\pm$ the *wave-automorphisms*, which correspond to Møller's wave-operators of scattering theory in Hilbert space. The object corresponding to the scattering operator is simply $S = \Omega_-^{-1}\Omega_+$ which we call here the *S-automorphism*. Its all-important property is that it is a convex automorphism of $\mathscr{S}$.

References

Jauch, J. M. (1958). *Helvetica Physica Acta*, **31**, 127.
Jauch, J. M. (1968a). *Foundations of Quantum Mechanics*, p. 68. Addison-Wesley, Reading, Mass.
Jauch, J. M., Misra, B. and Gibson, A. G. (1968b). *Helvetica Physica Acta*, **41**, 513.
Kronfli, N. S. (1969). Birkbeck College preprint. (Presented for publication.)
Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Vol. I, p. 105. Van Nostrand, Princeton, N.J.